Journal of Statistical Physics, Vol. 21, No. 6, 1979

Boundary Conditions and Cluster Property in Two-Dimensional Ising Ferromagnets

D. Merlini[1]

Received June 5, 1979

Using the Sherman theorem on paths, the cluster property, and the second GKS inequality, we obtain some results in favor of the non-existence of non-translation-invariant equilibrium states for two-dimensional Ising models with ferromagnetic short-range interactions in the low-temperature region. With a constraint on the interaction strength at the boundary, we prove that for the two-dimensional Ising model, all boundary conditions yield the unique translation-invariant correlation functions $\langle\sigma_X\rangle_{\mathbb{Z}^2,+}$, for $|X|$ even.

KEY WORDS: Ising ferromagnet; boundary conditions; cluster property; Sherman theorem on paths.

1. INTRODUCTION

It is well established[1] that in the three-dimensional Ising ferromagnet, in zero field, there exist non-translation-invariant equilibrium states, i.e., there exists at least a well-defined sequence of increasing cubes $\{\Lambda_i\}$, $i = 1, 2,..., \infty$, with an appropriate sequence of boundary conditions $\{b_{i,+-}\}$, such that, in the thermodynamic limit and at low enough temperatures, the equilibrium state so obtained is not translation invariant. This fact is related to and expresses the property that at low temperature the interface between the two pure phases is rigid, a result very different from the one obtained in the two-dimensional case,[2] where the oscillation of the interface is relatively large. In connection with this last result and a conjecture,[3] it is then believed that non-translation-invariant equilibrium states should not exist in the two-dimensional case. In other words, for all subset $X \subset \Lambda$, $|X|$ even, of points inside the box Λ, and any boundary condition b, $\langle\sigma_X\rangle_{\Lambda,b}$ should

Supported by the Fonds National Suisse de la Recherche Scientifique.

[1] Department of Physics, College of William and Mary, Williamsburg, Virginia.

0022-4715/79/1200-0739$03.00/0

approach $\langle \sigma_X \rangle_{\Lambda,+}$ as $|\Lambda| \to \infty$; $\langle \sigma_X \rangle_{\Lambda,+}$ denotes, as usual, the equilibrium state defined by means of plus boundary conditions, where all spins on the boundary are fixed in the configuration $\sigma = +1$. (Notice that in this work we are concerned with even correlation functions; if $|X|$ is odd, the connected problem is to show that $\langle \sigma_X \rangle_{\Lambda,b}$ should approach $\alpha \langle \sigma_X \rangle_{\Lambda,+} + (1 - \alpha)\langle \sigma_X \rangle_{\Lambda,-}$, $0 \leqslant a \leqslant 1$, as $|\Lambda| \to \infty$.)

A proof of the above conjecture is, to the best of our knowledge, still lacking and has not been completely ruled out. This paper is no exception; partial results have been obtained recently by different methods using refined inequalities.[4] Here we still reinforce the above conjecture and obtain partial results which apply to a large set of boundary conditions, some of which are not necessarily covered by the partial results obtained in the above reference. Moreover, with a constraint on the interaction strength at the boundary, it is proved here that if $|X|$ is even, all boundary conditions yield the unique translation-invariant correlation function $\langle \sigma_X \rangle_{\mathbb{Z}^2,+}$, defined by means of plus boundary conditions. Technically, the results apply also with minor modifications, to the less interesting, but nevertheless instructive case where the model is defined on domains Λ which are not subsets of $\mathbb{Z}^2$, and where the constraint on the boundary interaction strength may be removed; uniqueness of the even correlation functions for the model in cylindrical geometry at low temperatures is then obtained.[5]

The method is not restricted to the two-dimensional case and applies to ν-dimensional ferromagnets with short-range interactions. The results of the method are nevertheless more suggestive in one and two dimensions, since the cardinality of the boundary may grow at most as the linear dimension of the lattice, i.e., $|b| \leqslant c \cdot L$, $\forall b$.

The strategy of this work was advanced earlier[6] and the motivation of this paper was an attempt to prove the above conjecture by use of the duality transformation method, equivalent to low-temperature technique, and by use of the Sherman theorem on paths, specific to the two-dimensional Ising model.[7–10] We note that the use of such a theorem is not strictly necessary here; it nevertheless allows us to derive very simply a cluster property for even correlation functions without employing the more refined general Γ^* technique of the algebraic method[11]; in fact, using a suitable cluster expansion for low-temperature contours,[12,13] we may recover the Sherman theorem on paths in defining a mapping of the connected graphs (appearing in the Γ^* cluster expansion) into the set of closed contours or cycles.[14]

Boundary conditions and the Sherman theorem on paths are recalled in Section 2, while our partial results are derived and illustrated with some remarks in Section 3.

2. BOUNDARY CONDITIONS AND THE SHERMAN THEOREM ON PATHS

Let $A \subset \mathbb{Z}^2$ be a finite square box of $|\Lambda|$ sites. Let $\sigma_i = \pm 1$, $i \in \Lambda$, be an Ising spin variable and $\sigma_A = \prod_{i \in A} \sigma_i$, $A \subset \Lambda$, and let B be any bond on the lattice (B = two-point nearest neighbor subset for the Ising model). The analysis will be restricted to even correlation functions $\{\langle \sigma_X \rangle\}$, $|X|$ even, $X = (B_1, B_2, B_3, B_4, ..., B_n)$, near the center of the lattice. With $\mathscr{B}$ the set of all boundary conditions on the boundary $\partial\Lambda$ ($\Lambda \cap \partial\Lambda = \varnothing$), on the finite, fixed square box Λ, let each element $b \in \mathscr{B}$ be the set of sites $\{x_{i0}\} \subset \partial\Lambda$ where $\sigma_{x_{i0}} = -1$; the other points $x_{i0} \in \partial\Lambda$ are fixed in the configuration $\sigma = +1$. Let for each bond $B \subset \Lambda$, $\mu_B = \exp(-2K_B\sigma_B)$, $K_B = \beta J_B$, and for each $x_{i0} \in b$, $B_i = (x_i, x_{i0})$, such that $\sigma_{B_i} = \sigma_{x_i} \cdot \sigma_{x_{i0}} = -\sigma_{x_i}$; let $\mu_{B_i} = \exp(-2K_{B_i}\sigma_{x_i})$. The correlation function $\langle \sigma_X \rangle_{\Lambda,b}$, with boundary condition $b \in \mathscr{B}$, is given by

$$\langle \sigma_X \rangle_{\Lambda,b} = \frac{\langle \sigma_X \prod_i \exp(2K_{B_i}\sigma_{B_i}) \rangle_{\Lambda,+}}{\langle \prod_i \exp(2K_{B_i}\sigma_{B_i}) \rangle_{\Lambda,+}} = \frac{\langle \sigma_X \prod_i \mu_{B_i} \rangle_{\Lambda,+}}{\langle \prod_i \mu_{B_i} \rangle_{\Lambda,+}} \tag{1}$$

For later use, we recall that, with each X, $|X|$ even, inside Λ we may associate a path $P = (\bar{B}_1, \bar{B}_2, ..., \bar{B}_n)$ (a set of bonds inside Λ) such that $\sigma_X = \prod_{i=1}^{n} \sigma_{\bar{B}_i}$. The Sherman theorem on paths,[8] which we shall use later to derive a cluster property, holds in particular for a finite square Λ^*, with open boundary conditions and interaction energy $\{K_{B^*}^*\}$, ($K_{B^*}^*$ real) with $|\text{th } K_{B^*}^*| < 1$, $\forall B^* \subset \Lambda^*$, which may be different for each bond B^*; we use here the superscript $*$, since in applying the Sherman theorem we shall work with high-temperature closed graphs associated with the lattice Λ^* dual to Λ,[6] and such that for any pair of dual bonds B, B^*, th $K_{B^*}^* = e^{-2K_B}$. (Notice that closed graphs on the open square Λ^* coincide with low-temperature contours on Λ with plus boundary conditions.)

The theorem states that the reduced partition function $\tilde{Z}_{\Lambda^*}(\{K_{B^*}^*\})$ is given by the exponential of a sum of trajectories, i.e.,

$$\tilde{Z}_{\Lambda^*}(\{K_{B^*}^*\}) = \frac{Z_{\Lambda^*}(\{K_{B^*}^*\})}{2^{\Lambda^*} \prod_{B^* \subset \Lambda^*} \cosh K_{B^*}^*} = \exp\left[\sum_C \frac{(-1)^{N_C}}{\mu(C)} \prod_{B^* \in C} (\text{th } K_{B^*}^*)^{n_{B^*}}\right]$$
$$= \exp\left[\sum_C W(C)\right] \tag{2}$$

In (2), C denotes any cycle, or closed connected path on the lattice Λ^*, by weight $W(C)$; $\mu(C)$ is the multiplicity of the cycle C, N_C the number of self-crossings of C, and n_{B^*} the number of times a bond B^* occurs in C. Further, $\sum_{B^* \in C} n_{B^*} = l(C)$ is the length of C. Notice that a change of π

degrees in the trajectory to construct the path C is not allowed. We now derive our result.

3. PARTIAL RESULTS

It may be stated as follows: Let Λ be a square box of area $|\Lambda| = L^2$ and $b \in \mathscr{B}$, with $|b|$ the cardinality of b (from the symmetry property it is sufficient to consider b such that $|b| \leqslant 2L$); let l_b be the smallest distance between any point of b and X; $K = \beta J$ is the interaction strength $\forall B \subset \Lambda$ and $\{K_{B_i}\}$ is the interaction strength at the boundary (boundary fields). We then have the following result:

Theorem. If

$$4 \sum_{i=1}^{|b|} K_{B_i} + 2l_b(\ln 3 - 2K) \to -\infty \qquad \text{as } L \to \infty$$

then at low enough temperature $\langle \sigma_X \rangle_{\Lambda, b}(K, \{K_{B_i}\})$ converges to $\langle \sigma_X \rangle_{\mathbb{Z}^2, +}(K)$ as $|\Lambda| \to \infty$. Therefore all such boundary conditions yield the same translation-invariant correlation functions $\langle \sigma_X \rangle_{\mathbb{Z}^2, +}(K)$ defined by means of plus boundary conditions.

Proof. We recall that, if $\xi = \langle \sigma_X \rangle_{\Lambda, b} / \langle \sigma_X \rangle_{\Lambda, +}$, $\xi \leqslant 1$, $\forall b \in \mathscr{B}$. We now express the function $\mu_{B_i} = \cosh 2K_{B_i} - \sigma_{x_i} \sinh 2K_{B_i}$ in terms of the $\{\sigma_{x_i}\}$ and thus express ξ as a ratio of two sums involving expectation values in the pure phase, i.e.,

$$\begin{aligned} \xi &= \frac{\sum_{A \subset B} \alpha_A (-1)^{|A|} \langle \sigma_X \sigma_A \rangle_{\Lambda, +}}{\sum_{A \subset B} \alpha_A (-1)^{|A|} \langle \sigma_A \rangle_{\Lambda, +} \langle \sigma_X \rangle_{\Lambda, +}} \\ &= 1 + \frac{\sum_{A \subset B} \alpha_A (-1)^{|A|} (\langle \sigma_X \sigma_A \rangle_{\Lambda, +} - \langle \sigma_X \rangle_{\Lambda, +} \langle \sigma_A \rangle_{\Lambda, +})}{\sum_{A \subset b} \alpha_A (-1)^{|A|} \langle \sigma_X \rangle_{\Lambda, +} \langle \sigma_A \rangle_{\Lambda, +}} \end{aligned} \tag{3}$$

where by definition $\alpha_A = \alpha_A(\{K_{B_i}\})$. Applying the second GKS inequality $\langle \sigma_X \sigma_A \rangle_{\Lambda, +} \geqslant \langle \sigma_X \rangle_{\Lambda, +} \langle \sigma_A \rangle_{\Lambda, +}$, we obtain

$$\xi \geqslant 1 - \left[\max_{\substack{A \subset b \\ |A| \text{odd}}} \left(\frac{\langle \sigma_X \sigma_A \rangle_{\Lambda, +}}{\langle \sigma_X \rangle_{\Lambda, +} \langle \sigma_A \rangle_{\Lambda, +}} - 1 \right) \right] \frac{\langle \exp(2 \sum_i K_{B_i} \sigma_{x_i}) \rangle_{\Lambda, +}}{\langle \exp(-2 \sum_i K_{B_i} \sigma_{x_i}) \rangle_{\Lambda, +}} \tag{4}$$

Thus

$$\xi \geqslant 1 - \left(\max_{A \subset b} \frac{\langle \sigma_X \sigma_A \rangle_{\Lambda, +}}{\langle \sigma_X \rangle_{\Lambda, +} \langle \sigma_A \rangle_{\Lambda, +}} - 1 \right) \exp\left(4 \sum_i K_{B_i} \right) \tag{5}$$

where in (4) we used the crude lower and upper bounds

$$\left\langle \exp\left(\mp 2 \sum_i K_{B_i} \sigma_{x_i} \right) \right\rangle_{\Lambda, +} \gtrless \exp\left(\mp 2 \sum_i K_{B_i} \right).$$

The cluster property in the pure phase yields upper bounds of the form

$$\langle \sigma_X \sigma_A \rangle_{\Lambda,+} / \langle \sigma_X \rangle_{\Lambda,+} \langle \sigma_A \rangle_{\Lambda,+} \leqslant \exp[\alpha (e^{-2K})^{d_{X,A} \cdot \gamma}]$$

with α and γ some positive coefficients[12]; the method is general; we derive here such a bound, applying directly the Sherman theorem on paths, after a low–high temperature duality transformation[6]; by such a transformation, plus boundary conditions are mapped into open boundary conditions and for each pair of dual bonds (B, B^*), K^* is such that $\operatorname{th} K^*_{B^*} = e^{-2K_B}$. Then

$$C_{A^*} = \frac{\langle \sigma_X \sigma_A \rangle_{\Lambda,+}}{\langle \sigma_X \rangle_{\Lambda,+} \langle \sigma_A \rangle_{\Lambda,+}} (K) = \frac{\langle \mu^*_{X^*} \mu_{A^*} \rangle_{\Lambda^*,\mathrm{op}}}{\langle \mu^*_{X^*} \rangle_{\Lambda^*,\mathrm{op}} \langle \mu_{A^*} \rangle_{\Lambda^*,\mathrm{op}}} (K^*) \tag{6}$$

The above expectation values are to be computed in the dual system on Λ^* (open square), with interaction $K^*_{B^*}$, and for any dual path Y^* to Y, $Y^* = (B_1{}^*, B_2{}^*, ..., B_n{}^*)$,

$$\mu_{Y^*} = \exp\left(-2 \sum_i K^*_{B_i{}^*} \sigma_{B_i{}^*}\right) = \prod_{B_i{}^* \in Y^*} \mu^*_{B_i{}^*}$$

C_A appears as a ratio between the product of two partition functions, i.e., by definition $[X^* = (\bar{B}_1{}^*, \bar{B}_2{}^*, ..., \bar{B}_n{}^*);\ A^* = (B_1{}^*, B_2{}^*, ..., B^*_{|A^*|})]$,

$$C_{A^*} = \frac{\tilde{Z}_{\Lambda^*}(-K^*_{\bar{B}_i{}^*}, -K^*_{B_i{}^*}) \tilde{Z}_{\Lambda^*}(+K^*_{\bar{B}_i{}^*}, +K^*_{B_i{}^*})}{\tilde{Z}_{\Lambda^*}(-K^*_{\bar{B}_i{}^*}, +K^*_{B_i{}^*}) \tilde{Z}_{\Lambda^*}(+K^*_{\bar{B}_i{}^*}, -K^*_{B_i{}^*})} \tag{7}$$

since each $\mu^*_{B^*}$ changes the interaction from ferromagnetic to antiferromagnetic in the respective Hamiltonians. Applying the Sherman theorem to the above ratio, we see that the bulk contribution cancels exactly and only cycles C passing simultaneously through a subset of A^* of the dual path b^* (corresponding to the boundary b) and of X^* need to be considered; explicitly,

$$C_{A^*} = \exp\left[\sum_{\substack{C \cap X^* \neq \varnothing \\ C \cap A^* \neq \varnothing}} W(C) \kappa_C\right] \tag{8}$$

where

$$\kappa_C = (-1)^{|C \cap X^*| + |C \cap A^*|} + 1 - (-1)^{|C \cap X^*|} - (-1)^{|C \cap A^*|}$$

and

$$W(C) = \frac{(-1)^{N_C}}{\mu(C)} \prod_{B^* \in C} (\operatorname{th} K^*_{B^*})^{n_{B^*}} = \frac{(-1)^{N_C}}{\mu(C)} \prod_{B^* \in C} (e^{-2K_B})^{n_{B^*}}$$

$|C \cap X^*|$ and $|C \cap A^*|$ denote, respectively, the number of bonds of X^* and of A^* which occur with its multiplicity in C. Then a crude estimate as an upper bound is given for any A^* by

$$C_{A^*} = \exp\left[4 \sum_{\substack{C \cap X^* \neq \varnothing \\ C \cap A^* \neq \varnothing}} |W(C)|\right] \tag{9}$$

As in the Peierls argument, the number of cycles of length l containing a given bond is smaller than 3^l; from the definition of l_b we obtain

$$C_{A^*} \leqslant \exp\left[(3 \exp -2K)^{2l_b} \frac{1}{1 - 3\exp(-2\min_i K_{B_i})}\right] \tag{10}$$

with $\exp(-2\min_i K_{B_i}) < \frac{1}{3}$; then with (5) and (10), as $|\Lambda| \to \infty$ we have

$$\xi \geqslant 1 - \left[\exp\left(4 \sum_i K_{B_i}\right)\right](3 \exp -2K)^{2l_b \cdot r} \qquad (r > 0) \tag{11}$$

If $4 \sum_i K_{B_i} + 2l_b(\ln 3 - 2K) \to -\infty$, we obtain $\xi \geqslant 1$; since $\xi \leqslant 1$, $\xi = 1$, thereby proving the theorem.

Our result is now illustrated in the form of some remarks.

Remark 1. If the boundary fields are such that $K_{B_i} = K$, $\forall_i$, we obtain $\xi = 1$ for boundary conditions b such that $|b|/l_b < 1 - (\ln 3)/2K$, in the low-temperature region such that $e^{-2K} \leqslant 1/3^\alpha$, $\alpha > 1$; it is easily found that there exists b for which $\xi = 1$ from (11) with the two properties: (a) the inequality $\sigma_{x_{i0}} + \sigma_{\bar{x}_{i0}} \geqslant 0$ (or $\leqslant 0$) for all pairs of symmetric boundary points (with respect to a vertical axis in the middle of the box Λ) is not satisfied. (b) $|b|$ may satisfy the inequality $|b|/(4L - |b|) > 3/5$; and for such b, the partial result derived in Ref. 4 (and the reference therein) does not apply.

Remark 2. Let us impose a constraint on the interaction energy at the boundary, or strength of the boundary fields, i.e., let first $K_{B_i} = K'$, $\forall i$; then our result says that *all* boundary conditions b give $\xi = 1$, provided that $\ln 3 < 2K' < (2K - \ln 3)/4$. Moreover,

$$\lim_{|\Lambda| \to \infty} \langle \sigma_X \rangle_{\Lambda, b}(K, K') = \langle \sigma_X \rangle_{\mathbb{Z}^2, +}(K, K)$$

since from the second GKS inequality

$$\lim_{|\Lambda| \to \infty} \langle \sigma_X \rangle_{\Lambda, \mathrm{op}}(K, 0) \leqslant \lim_{|\Lambda| \to \infty} \langle \sigma_X \rangle_{\Lambda, +}(K, K') \leqslant \lim_{|\Lambda| \to \infty} \langle \sigma_X \rangle_{\Lambda, +}(K, K)$$

and $\langle \sigma_X \rangle_{\mathbb{Z}^2, \mathrm{op}}(J) = \langle \sigma_X \rangle_{\mathbb{Z}^2, +}(J)$, since there are only two translation-invariant states in the Ising ferromagnet.[17-20] Thus the correlation functions are insensitive to a change of the interaction strength at the boundary. It is believed that the constraint on the strength of the boundary fields on J' appears more for technical reasons and may be removed, at least in some cases, by a suitable choice of the domain Λ; in fact, we note the following.

Remark 3. Our results also apply to the Ising model defined on a set Λ not necessarily a subset of $\mathbb{Z}^2$; in particular, a refinement of the method allows us to remove the above constraint, yielding the uniqueness of the even correlation functions for the two-dimensional Ising model in cylindrical geometry at low temperature.[5]

Note: After this work was completed the author learned that a general proof for the non-existence of non-translation-invariant states in the two-dimensional Ising model was obtained by M. Aizenman and Y. Higuchi.

ACKNOWLEDGMENTS

The author wishes to thank J. Bricmont, J. L. Lebowitz, and C. Pfister for many stimulating and useful discussions on the subject, and J. L. Lebowitz for his kind hospitality. The author also acknowledges the hospitality of the Aspen Center for Physics during the summer of 1978, where part of this work was done.

REFERENCES

1. R. L. Dobrushin, *Theory Appl.* **17**:582 (1972).
2. G. Gallavotti, *Comm. Math. Phys.* **27**:103 (1972).
3. R. L. Dobrushin, *Teor. Mat. Fiz.* **12**:115 (1972).
4. A. Messager and S. Miracle-Sole, *J. Stat. Phys.* **17**:245 (1977) and references therein.
5. D. Merlini, preprint (1978), presented at the 41st Statistical Mechanics Meeting, Rutgers, May 1979.
6. C. Gruber, A. Hintermann, and D. Merlini, in *Lecture Notes in Physics*, No. 60 (Springer-Verlag, 1977), Chapter 5 and p. 159; see also Ref. 145.
7. S. Sherman, *J. Math. Phys.* **8**:399 (1960).
8. S. Sherman, *J. Math. Phys.* **4**:1213 (1963).
9. P. N. Burgoyne, *J. Math. Phys.* **4**:1320 (1963).
10. C. Gruber, EPFL-Lausanne, Cour AVCP, Verbier (Vd) (1972).
11. D. Ruelle, *Statistical Mechanics* (Benjamin, 1969), Chapter 4, p. 86.
12. G. Gallavotti, Martin-Löf, and S. Miracle-Sole, in *Statistical Mechanics and Mathematical Problems* (Battelle-Seattle 1971, Rencontres, Springer, New York, 1973).
13. G. Gallavotti and S. Miracle-Sole, *Comm. Math. Phys.* **7**:274 (1968).
14. D. Merlini, EPFL-Lausanne, unpublished (1973).
15. R. B. Griffiths, *J. Math. Phys.* **8**:478 (1967).
16. D. G. Kelley and S. Sherman, *J. Math. Phys.* **9**:466 (1969).
17. J. L. Lebowitz, *J. Stat. Phys.* **17**:245 (1977).
18. J. L. Lebowitz, *J. Stat. Phys.* **16**:463 (1977).
19. J. L. Lebowitz, in *Quantum Dynamics: Models and Mathematics* (Acta Physica Austriaca Suppl. 16, Springer, 1976), pp. 201–220.
20. G. Gallavotti and S. Miracle-Sole, *Phys. Rev. B* **5**:2555 (1972).